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Moduli of Smoothness Using Discrete Data

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1. INTRODUCTION

The r moduli of smoothness of a function are defined in terms of the differences $\Delta_h f(x) \equiv f(x+h) - f(x)$ and $\Delta_h^r f(x) \equiv \Delta_h(\Delta_h^{r-1} f(x))$ by $\omega_r(f, t) = \sup_{x, h \leqslant t} |\Delta_h^r f(x)|$. For continuous functions $f(x)$ on $[a, b]$, DeVore [4, p. 253] proved that $\sup_x |\Delta_{h_n}^2 f(x)| \leqslant M h_n^\alpha$, for some $\alpha < 2$, and a fixed sequence $\{h_n\}$ satisfying $h_n = o(1)$ and $1 \leqslant (h_n/h_{n+1}) \leqslant M$ implies $|\Delta_h^2 f(x)| \leqslant M h^\alpha$, for all h and x (such that $[x, x+2h] \subset [a, b]$). For $L_p(R)$ or $L_p(R^+)$, $1 \leqslant p \leqslant \infty$, a similar theorem was proved by the author [5] where $\|\Delta_h^2 f\|_{L_p}$ replaces the corresponding expression in L_∞ norm. Freud [7] proved for $L_2(T)$ ($T = [-\pi, \pi]$ for periodic fucntions) that $\|\Delta_{h_n}^r f\|_{L_2} = O(h_n^\alpha)$, for a sequence h_n as above, implies $\|\Delta_h^r f\|_{L_2} = O(h^\alpha)$. This theorem was generalized to $L_p(R)$ and $L_p(T)$ by Boman [1], and recently, the theorem was proved by Totik [12] for $L_p[a, b]$, $1 \leqslant p \leqslant \infty$.

The condition $1 \leqslant (h_n/h_{n+1}) \leqslant K$ is necessary, as there are examples to show that if $1 \leqslant (h_n/h_{n+1}) \leqslant K$ fails, the implication is no longer valid. For functions in $C(R)$ the condition was imposed on $|\Delta_{h_n}^r f(x)|$, for all x (and natural analogous restrictions were imposed in other spaces). It will be our goal to show how this condition can be relaxed and how we can replace the information on $|\Delta_{h_n}^r f(x)|$ by information on $|\Delta_{h_n}^r f(\xi_n + k h_n)|$, with ξ_n any fixed sequence of reals, h_n a fixed sequence with the conditions above, and $k = 0, \pm 1, \pm 2, \ldots$. In fact the sequence ξ_n is an added degree of freedom that we have in some cases and not an added requirement. In general, an added requirement is imposed on $f(x)$ except in the cases $h_n = K l^{-n}$ and $\xi_n = a$. In the case $h_n = 2^{-n}$, $\xi_n = 0$, and $f \in C[0, 1]$, the equivalence between $\sup_{0 \leqslant i \leqslant 2^n - r} |\Delta_{2^{-n}}^r f(i/2^n)| \leqslant M 2^{-n\alpha}$ and $\|\Delta_h^r f\|_{C[0, 1-rh]} \leqslant M_1 h^\alpha$ was established by Ciesielski [3] using orthogonal spline systems.

For proving that $|\Delta_{h_n}^r f(k h_n)| = O(h_n^\alpha)$ where $[k h_n, (k+r) h_n] \subset [a, b]$

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implies $|\Delta_h^r f(x)| \leqslant Mh^\alpha$ where $[x, x+rh] \subset [a, b]$, we will have to make a concession on h_n; we assume $h_n = Kl^{-n}$ with $\xi_n = a$ and $K = (b-a)/j$ (where j is an integer). This is needed in our proof that involves an extension theorem. That is, we construct a function $g(x)$ on a bigger interval which is equal to $f(x)$ in the given interval and for which $|\Delta_{h_n}^r g(kh_n)|$ everywhere is of the same order of magnitude as $|\Delta_{h_n}^r f(kh_n)|$, for those k for which $f(kh_n)$ is defined.

In order to facilitate our proof we will prove an approximation theorem about a specific Cardinal B spline approximation that will satisfy estimates on its rate of convergence and on its derivatives.

2. Cardinal B Spline Approximation Using Discrete Data

In order to prove our main theorem we will need an approximation theorem of a certain type. To introduce our approximation operator we reall the B spline of order k with equidistant knots supported by $[-(k/2)h, (k/2)h]$ given by $N(k, h, t) \equiv N(k, t/h)$ and $N(k, t) \equiv G_k(t) \equiv G_1 * G_{k-1}(t) \equiv \int_{-\infty}^{\infty} G_1(x)\, G_{k-1}(t-x)\, dx$, where $G_1(t) = 1$ for $|t| \leqslant \frac{1}{2}$ and $G_1(t) = 0$ elsewhere. The Schoenberg variation diminishing B spline with equidistant knots starting at ξ is given by

$$S_h(\xi, k, f; t) \equiv \sum_i f(ih+\xi)\, N(k, h, t-ih-\xi). \tag{2.1}$$

It is well known that $S_h(\xi, f, k; t)$ is of norm 1 as an operator on $C(R)$, which for any h is the identity on the functions 1 and x. It is an approximation operator using data at the points $ih + \xi$. We also have:

Lemma 2.1. *For a polynomial* $p(x)$ *of degree* $m \leqslant k-1$, $S_h(\xi, k, p; t) - p(t) = q(t)$, *where* $q(t)$ *is a polynomial of degree* $m-2$ *or* $q(t) = 0$ *if* $m - 2 < 0$.

Proof. We may choose $p(t) = t^m$. We observe that by (see [2, p. 138] or [8, p. 136]) using the formula for derivatives of B splines, we have

$$\begin{aligned}
&\frac{d}{dt} S_h(\xi, k, f; t)\\
&\quad = \frac{1}{h} \sum_i f(ih+\xi) \{ N(k-1, h, t-ih-\xi) - N(k-1, h, t-(i-1)h-\xi) \}\\
&\quad = \frac{1}{h} \sum_i \Delta_h f((i-1)h+\xi)\, N(k-1, h, t-ih-\xi)
\end{aligned}$$

and therefore

$$\frac{d^m}{dt^m} S_h(\xi, k, f; t) = \frac{1}{h^m} \sum \Delta_h^m f((i-m)h+\xi) N(k-m, t-ih-\xi),$$

which is valid at every point except in the case $m = k-1$, when the derivative above is valid only for $t \neq \xi + ih$ for any integer i. Recalling that $\Delta_h^m t^m = m!h^m$ and that $m-1$ derivatives of $S_h(\xi, k, f; t)$ are continuous if $m \leqslant k-1$, we have $(d/dt)^m p(t) - (d/dt)^m S_h(\xi, k, p; t) = 0$, and we have proved that $q(t)$ is of degree $m-1$ or $q(t) = 0$ if $m-1<0$. Using the linearity of $S_h(\xi, k, f; t)$, the fact that for $f(x) = (x-\xi)^m$, $S_h(\xi, k, f, t)$ is odd or even as a function of $(t-\xi)$ if m is an odd or an even integer, and the results that we have already proved for smaller integers than m, we complete the proof.

We are now in a position to define the general approximation operator $S_{h,r}$:

$$S_{h,r}(\xi, k, f; t) = \sum_{l=1}^{r} (-1)^{l+1} \binom{r}{l} S_h^l(\xi, k, f; t), \tag{2.2}$$

where $S_n^l(\xi, k, f; t) \equiv S_h(\xi, k, S_h^{l-1}; t)$ and $S_h^1(\xi, k, f; t) \equiv S_h(\xi, k, f; t)$.

LEMMA 2.2. *As an operator on $C(R)$, we have*

$$\| S_{h,r}(\xi, k, f; t) \| \leqslant 2^r - 1, \tag{2.3}$$

and for a polynomial $p(t)$ of degree m, $m < \min(2r, k)$,

$$S_{h,r}(\xi, k, p; t) - p(t) = 0. \tag{2.4}$$

Proof. Using the definition of $S_{h,r}$ and $\| S_h \| \leqslant 1$ as an operator from $C(R)$ to $C(R)$, we derive (2.3). Formula (2.4) is simply the r iterate of Lemma 2.1.

The desired approximation operator will be $S_{h,r}(\xi, k, f; t)$ for which with $k \geqslant 2r$ we will prove:

THEOREM 2.3. *For $k \geqslant 2r$, we have*

$$\| S_{h,r}(\xi, k, f, t) - f(t) \|_{C(R)} \leqslant K\omega_{2r}(f, h)$$

and locally

$$\| S_{h,r}(\xi, k, f; t) - f(t) \|_{C[a,b]} \leqslant K \sup_{\substack{|\eta| \leqslant h \\ a-Lh \leqslant x \leqslant b+Lh}} | \Delta_\eta^{2r} f(x) |,$$

where K and L depend on r and k but not on f, h, or ξ.

Remark. A formula that will recover polynomials of certain order and therefore from which a theorem like 2.3 is possible has been proved (see [7, p. 218]), but the data there about the function is not compatible with our requirement here.

Proof of Theorem 2.3. For $f \in C^{2r}$, we have

$$f(x)=f(t)+\frac{(x-t)}{1!}f'(t)+\cdots+\frac{(x-t)^{2r-1}}{(2r-1)!}f^{(2r-1)}(t)$$
$$+\frac{(x-t)^{2r}}{(2r)!}f^{(2r)}(\zeta(x,t));$$

as $S_{h,r}(\zeta,k,f,t)$ is a projection on polynomial of degree $<2r$, we have

$$|S_{h,r}(\xi,k,f,t)-f(t))$$
$$=\frac{1}{(2r)!}|S_{h,r}(\xi,k,(t-\cdot)^{2r}f^{(2r)}(\zeta(\cdot,t);t)|$$
$$\leqslant\frac{1}{(2r)!}\sum_{l=1}^{r}\binom{r}{l}\max_{\zeta\in I_l}|f^{(2r)}(\zeta)|\cdot S_h^l(\xi,k,(t-\cdot)^{2r},t),$$

where $I_l=[t-lkh/2,t+lkh/2]$.
Therefore, we have

$$|S_{h,r}(\xi,k,f,t)-f(t)|\leqslant\frac{1}{(2r)!}\sum_{l=1}^{r}\binom{r}{l}\left(\frac{lkh}{2}\right)^{2r}\max_{|\zeta-t|\leqslant rkh/2}|f^{(2r)}(\zeta)|.$$

We can now write

$$f_h(x)=\frac{(r/h)^{2r}}{\binom{2r}{r}}\int_{-h/2r}^{h/2r}\cdots\int_{-h/2r}^{h/2r}\sum_{l=0}^{r-1}(-1)^{r-l-1}\binom{2r}{l}$$
$$\times f(x+(r-l)(u_1+\cdots+u_{2r}))\,du_1\cdots du_{2r}.$$

The known estimates (similar to [11, p. 163]) are

$$|f(x)-f_h(x)|\leqslant\frac{1}{\binom{2r}{r}}\sup_{|\eta|\leqslant h}|\Delta_\eta^{2r}f(x-r\eta)|$$

and

$$\left|\left(\frac{d}{dx}\right)^{2r}f_h(x)\right|\leqslant\left(\frac{r}{h}\right)^{2r}\frac{1}{\binom{2r}{r}}\sum_{l=1}^{r}\binom{2r}{l}\sup_{|\eta|\leqslant h/r}|\Delta_{(r-l)\eta}^{2r}f(x-r(r-l)\eta)|$$
$$\leqslant h^{-2r}\left\{r^{2r}\frac{1}{\binom{2r}{r}}\sum_{l=1}^{r}\binom{2r}{l}\right\}\sup_{|\eta|\leqslant h}|\Delta_\eta^{2r}f(x-r\eta)|.$$

The above estimates for $f_h^{(2r)}$ and $f-f_h$ can be deduced from computation done elsewhere but are easy to deduce directly from the definition of f_h.

Combining the estimates of $|f-f_h|$ and $f_h^{(2r)}$ with the above, we get

$$|S_{h,r}(\xi, k, f; t)-f(t)| \leqslant |S_{h,r}(\xi, k, f-f_h; t)| + |f_h(t)-f(t)| + |S_{h,r}(\xi, k, f_h, t)-f_h(t)|$$

and complete the proof using Lemma 2.2 and the definition on the modulus of continuity. We can observe that

$$K \leqslant \left\{\frac{1}{(2r)!}\sum_{l=1}^{r}\binom{r}{l}\left(\frac{lk}{2}\right)^{2r}\right\}\left\{r^{2r}\frac{1}{\binom{2r}{r}}\sum_{l=1}^{r}\binom{2r}{l}\right\}+2^r\cdot\frac{1}{\binom{2r}{r}},$$

and that $L \leqslant (rk/2)+r$, but these constants are not important for later proofs.

3. The Main Theorem with the Restriction $f \in \mathrm{Lip}\,\beta$

The main theorem will be proved below for functions on R first.

Theorem 3.1. *For $f \in C(R)$ satisfying $|\Delta_h f(x)| \leqslant Kh^\beta$, for some β and K, the condition $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$, where $\alpha \leqslant m$ for all k, a sequence h_n satisfying $h_n = o(1)$ and $1 \leqslant (h_n/h_{n+1}) \leqslant M$, and some sequence ξ_n of reals implies $|\Delta_h^m f(x)| \leqslant K_1 h^\alpha$, for all x and all h.*

Proof. We choose r such that $2r \geqslant m+1$ and use the approximation operator $S_{h_n,r}(\xi_n, 2r, f; t)$. From Theorem 2.3,

$$\|S_{h_n,r}(\xi_n, 2r, f; t)-f(t)\| \leqslant M\omega_{2r}(f, h_n) \leqslant M\cdot 2^{2r-m}\omega_m(f, h_n). \quad (3.1)$$

Using the definition of $S_{h_n,r}$, we have

$$\left|\left(\frac{d}{dt}\right)^m S_{h_n,r}(\xi_n, 2r, f; t)\right| \leqslant \sum_{l=1}^{r}\binom{r}{l}\left|\left(\frac{d}{dt}\right)^m S_{h_m}^l(\xi_n, 2r, f; t)\right|.$$

Observing that

$$S_{h_n}^l(\xi_n, 2r, f, t) = \sum_i \left(\sum_j \alpha_j(l) f(\xi_n + (i+j)h_n)\right) N(2r, h_n, t-ih_n-\xi_n),$$

where $\alpha_j(l) \geqslant 0$, $\sum_j \alpha_j(l) = 1$, for $|j| \leqslant (l-1)r$, we have

$$\left(\frac{d}{dt}\right)^m S^l_{h_n}(\xi_n, 2r, f; t)$$
$$= h_n^{-m} \sum_i \left(\sum_j \alpha_j(l)\, \Delta^m_{h_n} f(\xi_n + (i-m+j)\, h_n) \right) N(2r-m, h_n, t - ih_n - \xi_n) \tag{3.2}$$

and therefore

$$\left| \left(\frac{d}{dt}\right)^m S^l_{h_n}(\xi_n, 2r, f; t) \right| \leqslant h_n^{-m} \sup_k |\Delta^m_{h_n} f(\xi_n + kh_n)|. \tag{3.3}$$

We have now

$$\Delta^m_h f(t) = \Delta^m_h(f(t) - S_{h_n}(\xi_n, 2r, f; t)) + \Delta^m_h S_{h_n}(\xi_n, 2r, f; t),$$

from which, using (3.1) and (3.3),

$$|\Delta^m_h f(t)| \leqslant 2^m M 2^{2r-m} \omega_m(f, h_n) + (2^r - 1)\left(\frac{h}{h_n}\right)^m \sup_k |\Delta^m_{h_n} f(\xi_n + kh_n)|$$

or

$$|\Delta^m_h f(t)| \leqslant A\omega_m(f, h_n) + B\left(\frac{h}{h_n}\right)^m h_n^\alpha. \tag{3.4}$$

We will show under the assumption on f that an equation of the type (3.4) for $\alpha \leqslant m$ implies that for $f(x) \in \operatorname{Lip} \beta$, i.e., $|\Delta^m_h f(t)| \leqslant Ch^\beta$, for some β, that $|\Delta^m_h f(x)| \leqslant K_1 h^\alpha$. One can choose a subsequence η_n of h_n which satisfies $1 < T_1 \leqslant (\eta_n/\eta_{n+1}) \leqslant T_2 < \infty$ (for T_1 and T_2 big enough). We now have for some n, $\eta_n < h \leqslant \eta_{n-1}$,

$$|\Delta^m_h f(t)| \leqslant A\omega_m(f, \eta_n) + B\left(\frac{h}{\eta_n}\right)^m \eta^\alpha$$
$$\leqslant A^s \omega_m(f, \eta_{n+s}) + B\left(\frac{h}{\eta_n}\right)^m \eta_n^\alpha + B \sum_{j=1}^{s-1} \left(\frac{\eta_{n+j-1}}{\eta_{n+j}}\right)^m A^j \eta^\alpha_{n+j}.$$

Choosing T_1 and then T_2 such that $A/T_1^\beta < 1$ and therefore $A/T_1^\alpha < 1$, we have $A^s \omega_m(f, \eta_{n+s}) \leqslant C(A/T_1^\beta)^s h^\beta \leqslant C(A/T_1^\beta)^s$.

For s big enough such that $A^s\omega_m(f, \eta_{n+s}) \leqslant M_1 h^\alpha$, we have

$$|\Delta_h^m f(t)| \leqslant M_1 h^\alpha + B\left(\frac{h}{\eta_n}\right)^m \eta_n^\alpha + B\sum_{j=1}^{s-1}\left(\frac{\eta_{n+j-1}}{\eta_{n+j}}\right)^m A^j \eta_{n+j}^\alpha$$

$$\leqslant M_1 h^\alpha + BT_2^{m-\alpha} h^\alpha + BT_2^m \sum_{j=1}^{s-1} A^j T_1^{-j\alpha} \eta_n^\alpha$$

$$\leqslant h^\alpha (M_1 + BT_2^{m-\alpha} + BT_2^m (1/1 - (A/T_1^\alpha))),$$

and the last expression does not depend on s. This completes the proof.

THEOREM 3.2. *For* $f \in \operatorname{Lip}\beta$ *in* $[a, b]$, *for some* $\beta > 0$ *(and therefore* $f \in C[a, b]$*),* $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$, *for some* α, $\alpha \leqslant m$, *and all* k *and* n *such that* $[\xi_n + kh_n, \xi_n + (k+m) h_n] \subset [a, b]$, *where the sequence* h_n *and* ξ_n *are those of Theorem 3.1, implies* $|\Delta_h^m f(x)| \leqslant K_1 h^\alpha$, *for all* h *and* x *satisfying* $[x, x+mh] \subset [a+Th, b-Th]$, *where* T *is independent of* f, h *(but* T *depends on* m*), and* K, *is that of Theorem 3.1.*

Remark. Actually Theorem 3.2 contains Theorem 3.1 but the present arrangement may help clarify the proof.

Proof of Theorem 3.2. We follow the proof of Theorem 3.1 with special care in some of the steps involving the domain supporting the approximation process. For the main inequality (3.1), we have

$$\sup_{t, t+mh \in [a_1, b_1]} |\Delta_h^m f(t)|$$

$$\leqslant A\omega_m(f, h_n, [a_1 - Lh_n, b_1 + Lh_n]) + B\left(\frac{h}{h_n}\right)^m$$

$$\times \sup\{|\Delta_{h_n}^m f(\xi_n + kh_n)|; [\xi_n + (k-2r)h_n, \xi_n + (k+m+2r)h_n] \subset [a_1, b_1]\},$$

where $\omega_m(f, h_n, [a_1 - Lh_n, a_2 + Lh_n]) = \sup_{\eta \in h_n}\{|\Delta_\eta^m f(x)|; [x, x+\eta m] \subset [a_1 - Lh_n, b_1 + Lh_n]\}$. We choose now η_n as in the proof of Theorem 3.1, and because of their geometric progression the result will be valid if $[a_1 - L_1\sum_{l=n}^\infty \eta_l, b_1 + L_1\sum_{l=n}^\infty \eta_l] \subset [a, b]$, for $L_1 = \max(L, 2r+1]$. Recall now that $L_1\sum_{l=n}^\infty \eta_l \leqslant L_1\eta_n \sum_{l=n}^\infty (1/T_1)^{n-l} = L_1(T_1/T_1 - 1)\eta_n = L_2\eta_n$. In other words, choosing $a_1 = a + 2L_2 h$ and $b_1 = b - 2L_2 h$ will do. The rest of the proof follows that of Theorem 3.1 exactly.

4. RELAXING THE CONDITION $f \in$ LIP β

In this section we will show how the a priori assumption $f \in \operatorname{Lip}\beta$, for some $\beta > 0$, can be proved by imposing a different condition on f or on the sequence. We will need first the next two lemmas.

LEMMA 4.1. *For* $f \in L_\infty(R)$, $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$ implies $|\Delta_{h_n} f(\xi_n + kh_n)| \leqslant AK^{1/n} h_n^{\alpha/n} \|f\|_{L_\infty}^{1-(1/n)}$, *where* A *is independent of* f, ξ_n, *and* h_n, *and therefore* $|\Delta_{h_n} f(\xi_n + kh_n)| \leqslant K_1 h_n^{\alpha/n}$.

Proof. This is a lemma on sequences in $l_\infty(Z)$ for the sequence $a_k = f(\xi_n + kh_n)$ and obtaining a Kolmogorov-type inequality there. In an earlier paper [6] estimates were given for the constant A which yield the existence of such constants.

LEMMA 4.2. *If* $f \in L_\infty[a, b]$, $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$, *where* $[\xi_n + kh_n, \xi_n + (k+m) h_n] \subset [a, b]$ *implies* $|\Delta_{h_n} f(\xi_n + kh_n)| \leqslant K_2 h_n^{\alpha_1/m}$, *where* $\alpha_1 = \min(\alpha, 1)$ *for* $[\xi_n + kh_n, \xi_n + (k+1) h_n] \subset [a, b]$ *and* $h_n \leqslant (b-a)/6m$.

Proof. First, we observe that Lemma 4.1 is valid on $[a, \infty)$ or $(-\infty, b]$ using the result of [6] for $l_\infty(N)$ rather than for $l_\infty(Z)$. To prove the result for $[a, b]$, we multiply f by $g_1(x)$, where $g_1(x) = 1$ for $a \leqslant x \leqslant (a+2b)/3$, $g_1(x) = 0$ for $(a+5b)/6 \leqslant x < \infty$, and is defined linearly elsewhere. For $\phi_1(x) \equiv f \cdot g_1$, $|\Delta_{h_m}^m \phi_1(\xi_n + kh_n)| \leqslant |\Delta_{h_n}^m f(\xi_n + kh_n)| + |\Delta_{h_n} g_1(\xi_n + kh_n)| C \|f\| \leqslant Kh_n^\alpha + (h_n/6(b-a)) C \|f\|$. ϕ_1 is now defined in $[a, \infty)$ and we obtain the estimate for $|\Delta_{h_n} \phi_1(\xi_n + kh_n)|$. Similarly, we can derive the estimate for $\phi_2 = f \cdot g_2$, where $g_2(x) = 1$ for $(2a+b)/3 \leqslant x \leqslant b$, $g_2(x) = 0$ for $-\infty < x \leqslant (5a+b)/6$, and is defined linearly elsewhere. Therefore $|\Delta_{h_n} \phi_i(\xi_n + kh_n)|$ is bounded by $C_1 h_n^{\alpha_1/m}$ and hence $|\Delta h_n f(\xi_n + kh_n)| \leqslant 2C_1 h_n^{\alpha_1/m}$.

Remark. For our needs it does not matter that α_1/m replaces α/m since all that matters is that the condition is valid for some $\beta > 0$.

LEMMA 4.3. *If* f *is locally monotonic, that is for some fixed* d *in any interval of length* d *there is at most one change of direction, then* $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$, for h_n satisfying $1 \leqslant h_n/h_{n+1} \leqslant M$, implies $|\Delta_h f(x)| \leqslant K_2 h^{\alpha/m}$. If the condition $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$ and local monotonicity is given in $[a, b]$, they imply $|\Delta_h f(x)| \leqslant K_2 h^{\alpha_1/m}$, for $[x, x+h] \subset [a, b]$, where $\alpha_1 = \min(1, \alpha)$.

Proof. We restrict ourselves to the case in which the conditions are given on R since in the other case the proof is similar. Using Lemma 4.1, we have $|\Delta_{h_n} f(\xi_n + kh_n)| \leqslant K_1 h_n^{\alpha/m} \equiv K_1 h_n^\beta$. We construct a subsequence of h_n, η_n satisfying $1 < T_1 < (\eta_n/\eta_{n+1}) \leqslant T_2 < \infty$, with $T_1 \geqslant 2$.

Given two points x and y, $x < y$, we may assume that there is no change of direction between them, otherwise (if there is a change of direction at ζ), we may write the estimate $|f(x) - f(y)| \leqslant \max(|f(x) - f(\zeta)|, |f(y) - f(\zeta)|)$. We estimate separately $|f(x) - f((x+y)/2)|$ and $|f(y) - f((x+y)/2)|$ (using the knowledge that f is monotonic around $(x+y)/2$. We choose n such that $\eta_n < (1/2)|x-y| \leqslant \eta_{n-1}$ and integers

$k(n)$ and $l(n)$ such that $x \leqslant \xi_n + k(n)\,\eta_n < (x+y)/2 \leqslant \xi_n + l(n)\,\eta_n$, and the integers $k(n)$ and $l(n)$ are minimal satisfying it. We have $|f(x) - f((x+y)/2)| \leqslant |l(n) - k(n)| \cdot K_1 \eta_n^\beta + |f(x) - f(\xi_n + k(n)\,\eta_n)| \leqslant ([T_2]+2)\,K_1 \eta_n^\beta + |f(x) - f(\xi_n + k(n)\,\eta_n)|$. Continuing the process with $[x, \xi_n + k(n)\,\eta_n]$ whose length is smaller than η_n, we have

$$\left| f(x) - f\left(\frac{x+y}{2}\right)\right| \leqslant (T_2+2)\,K_1 \sum_{r=n}^{\infty} \eta_r^\beta$$

$$\leqslant (T_2+2)\,K_1 \eta_n^\beta \sum_{r=1}^{\infty} 2^{-\beta r} \leqslant K_2\,|x-y|^\beta$$

which completes the proof.

More useful, in particular for Section 8, is the following lemma.

LEMMA 4.4. *If* $f(x)$ *is locally absolutely continuous and* $|\Delta_{h_n}^m f(\xi_n + kh_n)| \leqslant Kh_n^\alpha$, *for* $1 \leqslant h_n/h_{n+1} \leqslant M$, *then* $f \in \mathrm{Lip}(\alpha/m)$. *If the condition* $|\Delta_{h_n}^m f(\xi_n + kh_n)|$ *is given in* $[a, b]$, *then* $f \in \mathrm{Lip}(\alpha_1/m)$ *there.*

Proof. We will follow partly the proof of Lemma 4.3. We have $|\Delta_{h_n} f(\xi_n + kh_n)| \leqslant K_1 h_n^\beta$, for the k's in question and $\beta = \alpha_1/m$, and choose a subsequence η_n of h_n such that $1 < T_1 \leqslant \eta_n/\eta_{n+1} \leqslant T_2 < \infty$, where $T_1 \geqslant 3^{1/\beta}$. For $x < y$, we choose n such that $\eta_n < |x-y| \leqslant \eta_{n-1}$ and integers $k_1(n)$ and $k_2(n)$ such that $x \leqslant \xi_n + k_1(n)\,\eta_n < \xi_n + k_2(n)\,\eta_n \leqslant y$, where $k_1(n)$ is minimal and $k_2(n)$ is maximal satisfying the above. Obviously, $|f(x) - f(y)| \leqslant |f(x) - f(\xi_n + k_1(n)\,\eta_n)| + |k_2(n) - k_1(n)|\,K_1 \eta_n^\beta + |f(\xi_n + k_2(n)\,\eta_n) - f(y)|$. We have $|k_1(n) - k_2(n)| \leqslant T_2$ and $|x - \xi_n - k_1(n)\,\eta_n| + |y - \xi_n - k_2(n)\,\eta_n| < 2\eta_n$. We now find δ such that for $\sum_{i=1}^{l} |\zeta_i - z_i| < \delta$, $\zeta_i < z_i \leqslant \zeta_{i+1}$, we have $\sum_{i=1}^{l} |f(\zeta_i) - f(z_i)| < h^\beta$, where $h = |x-y|$. We continue the above process with the two intervals left. We obtain $|f(x) - f(y)| \leqslant T_2 K_1 (\eta_n^\beta + 2\eta_{n+i}^\beta) + I$, where I is a sum on four intervals of total length $4\eta_{n+i} \leqslant 4\eta_{n+1}$. In general, after l steps,

$$|f(x) - f(y)| \leqslant T_2 K_1 (\eta_2^\beta + \cdots + 2^l \eta_{n+l}^\beta) + I(l) \leqslant K_2 \eta_n^\beta + I(l),$$

where $I(l) = \sum_{i=1}^{2^l} |f(\zeta_i) - f(z_i)|$, $\sum |\zeta_i - z_i| \leqslant 2^{l+1} \eta_{n+l}$. Since $T_1 > 3^{1/\beta} > 3$, $\sum |\zeta_i - z_i| < 2^l 3^{-l} \eta_n$ and smaller than δ for sufficiently big l, we have $|f(x) - f(y)| \leqslant \{2T_2 K_1 + 1\}\, h^\beta$, where $h = |x-y|$.

LEMMA 4.5. *If* $\xi_n = \xi$ *and* $h_n = Bl^{-n}$, *for a fixed integer* l, $f \in C(R)$ *and* $|\Delta_h^m f(\xi + kh_n)| \leqslant Kh_n^\alpha$ *implies* $|\Delta_h f(x)| \leqslant K_2 h^{\alpha/m}$. *In case the assumption above is given in an interval, then* $|\Delta_h f(x)| \leqslant K_2 h^{\alpha_1/m}$ *there.*

Proof. We have to choose the points appropriately, but because $\xi_n = \xi$ and $h_n = Bl^{-n}$, we fit $\xi + kBl^{-n} = \xi + klBl^{-n-1}$, and therefore we have in the proof of the last lemma only two intervals left at any stage and therefore continuity is sufficient.

We are now able to deduce the following as corollaries.

THEOREM 4.6. *For $f(x)$ locally monotonic and continuous or $f(x)$ locally absolutely continuous, the implications of Theorem* 4.1 *and* 4.2 *are valid without the a priori assumption that $f(x) \in \text{Lip}\,\beta$.*

THEOREM 4.7. *For $\xi_n = \xi$ and $h_n = Kl^{-n}$, the implications of Theorems* 4.1 *and* 4.2 *are valid without the a priori assumption that $f(x) \in \text{Lip}\,\beta$.*

5. FURTHER CASES

In this section we will achieve as corollaries of the theorems in Sections 3 and 4 further results first on periodic functions and then for the second difference on an interval or on R^+.

THEOREM 5.1. *For $f \in C(T)$, continuous functions with period 2π, the condition $|\Delta_{h_n}^m f(kh_n)| \leqslant Kh_n^\alpha$, $\alpha \leqslant m$, $h_n = (2\pi/r)\,l^{-n}$ (with some integer r), implies $|\Delta_h^m f(x)| \leqslant K_1 h^\alpha$, for all x and h.*

Proof. We use the results from Theorems 3.2 and 4.7 on $(-\delta, 2\pi + \delta)$ and the periodicity to obtain this corollary.

For $m = 2$ and $f \in C(R^+)$, we can obtain the following result.

THEOREM 5.2. *Suppose h_n is as given in Theorem* 3.1, *$|\Delta_{h_n}^2 f(kh_n)| \leqslant Mh_n^\alpha$ and f satisfies one of the following conditions: (a) $f \in \text{Lip}\,\beta$ in R^+, (b) $f \in C(R^+)$ and $f(x)$ is locally monotonic, (c) $f(x)$ is absolutely continuous. Then $|\Delta_h^2 f(x)| \leqslant M_1 h^\alpha$, for all $x \in R^+$ and $h \geqslant 0$. If $h_n = Al^{-n}$, then instead of (a), (b), or (c), we just assume that $f \in C(R^+)$ and still $|\Delta_h^2 f(x)| \leqslant M_1 h^\alpha$, for all $x \in R^+$ and $h \geqslant 0$.*

Remark. Theorems 3.2, 4.6, and 4.7 would imply the validity of this theorem for $x \in [\delta, \infty)$ and $h > 0$.

Proof. We define $f(x)$ in $(-\infty, 0)$ as $f(0) - (f(-x) - f(0))$ and obtain, for all k, $|\Delta_{h_n}^2 f(kh_n)| \leqslant Mh_n^\alpha$ in R and therefore $|\Delta_h^2 f(x)| \leqslant M_1 h^\alpha$, for all x. But for $x \geqslant 0$ the new function coincides with the function in the theorem.

THEOREM 5.3. *For $f(x) \in C[0, 1]$ and $h_n = l^{-n}$, the condition $|\Delta_{h_n}^2 f(kh_n)| \leqslant Mh_n^\alpha$, for $k = 0, 1, \ldots, l^n - 2$ and all n, implies $|\Delta_h^2 f(x)| \leqslant M_1 h^\alpha$, for x, $x + 2h \in [0, 1]$.*

Proof. We define $g(x) = f(x) - f(0)(1 - x) - f(1)\,x$, *for* $x \in [0, 1]$,

$g(x) = -g(-x)$, for $x \in [-1, 0]$, and $g(x) = g(x+2k)$, elsewhere (see Timan [11]). The function g satisfies $|\Delta^2_{h_n} g(kh_n)| \leqslant Mh^\alpha_h$, and therefore $|\Delta^2_h g(x)| \leqslant Mh^\alpha$, for all x and h, but in $[0, 1]$, $|\Delta^2_h f(x)| = |\Delta^2_h g(x)|$.

For higher differences this theorem would be generalized with a substantial amount of work.

6. An Extension Theorem

In order to prove a result like Theorem 3.1 or Theorem 4.7 for differences of order $r > 2$ on an interval $[a, b]$ or R^+, we will need an extension theorem. An extension theorem would provide a function g identical with $f(x)$ inside the interval or in R^+, satisfying an estimate on $|\Delta^r_{h_n} g(kh_n)|$ similar to that on $|\Delta^r_{h_n} f(kh_n)|$ but in a bigger interval or in R. An extension theorem that uses derivatives rather than differences is the well-known Whitney extension theorem (see, for instance, [10]). For differences given at all points x, an extension theorem was achieved using Stekelov-type integrals and the Whitney extension theorem [9]. Here the proof would be different as only data on kh_n are used.

Theorem 6.1. *Suppose $f(x) \in C[0, A]$ for which $|\Delta^m_{2^{-n}} f(k2^{-n})| \leqslant M2^{-n\alpha}$, for $(k+m)2^{-n} \leqslant A$, where k is a positive integer and α is not an integer unless $\alpha = m$. Then we can construct $g(x) \in C[-A, A]$ such that $|\Delta^m_{2^{-n}} g(k2^{-n})| \leqslant K2^{-n\alpha}$, where K depends only on m, α, M, $B \equiv \sup_{k,n} |f(k2^{-n})|$ and A.*

Remark. Dependence on A stems only from our construction. This restriction could have been removed (with some additional work) but for our purposes there is no need for this improvement. It is the distinction between Lipschitz and Zygmund classes that causes the restriction "α is not an integer unless $\alpha = m$."

Proof. We first assume $m - 1 < \alpha \leqslant m$. Using the known result [11, p. 105], we have

$$\Delta^{m-1}_{2h} f(x) - 2^{m-1} \Delta^{m-1}_h f(x) = \sum_{\nu=0}^{m-2} \sum_{\mu=\nu+1}^{m-1} \binom{m-1}{\mu} \Delta^m_h f(x + \nu h).$$

We substitute $h = 2^{-n}$ and $x = 0$ and multiply both sides by $(2^{m-1})^{n-1}$ to get

$$\begin{aligned} &|2^{(m-1)(n-1)} \Delta^{m-1}_{2^{-n+1}} f(0) - 2^{(m-1)n} \Delta^{m-1}_{2^{-n}} f(0)| \\ &\leqslant 2^{(m-1)n} \frac{m-1}{2} \max |\Delta^m_{2^{-n}} f(k2^{-n})| \\ &\leqslant \frac{m-1}{2} M \cdot 2^{-n(\alpha - m + 1)}. \end{aligned}$$

From this behaviour we may deduce that $\lim_{n\to\infty} 2^{(m-1)n}\Delta^{m-1}_{2^{-n}} f(0) \equiv C_{m-1}$ exists. An estimate for C_{m-1} is given by $|C_{m-1}| \leqslant |2^{(m-1)(n-l-1)}\Delta^{m-1}_{2^{-n-l}} f(0)| + ((m-1)/2)\, M 2^{-(n-l)\alpha}\,(1/(1-2^{-\alpha+m-1}))$, and l is chosen so that $2^{-(n-l-1)}m < A$ or $|C_{m-1}| \leqslant K_{m-1}$, with K_{m-1} satisfying what K of our theorem is supposed to satisfy. We now examine in $[0, A]$ the function $g_{m-1}(x) = f(x) - (1/(m-1)!)\, C_{m-1} x^{m-1}$. Obviously, $\Delta^m_{1/2^n} g_{m-1}(k/2^n) = \Delta^m_{1/2^n} f(k/2^n)$ for $k = 0, 1, \ldots$. Moreover,

$$
\begin{aligned}
&|\Delta^{m-1}_{1/2^n} g_{m-1}(0)| \\
&\quad\leqslant 2^{-(m-1)n}\,|2^{(m-1)n}\Delta^{m-1}_{2^{-n}} f(0(- C_{m-1}| \\
&\quad\leqslant 2^{-(m-1)n} \sum_{l=n}^{\infty} |2^{(m-1)l}\Delta^{m-1}_{2^{-l}} f(0) - 2^{(m-1)(l+1)}\Delta^{m-1}_{2^{-l+1}} f(0)| \\
&\quad\leqslant 2^{-(m-1)n} \sum_{l=n}^{\infty} \frac{m-1}{2}\, M 2^{-(l+1)(\alpha-m+1)} \\
&\quad\leqslant K_{m-1}(1)\, 2^{-n\alpha},
\end{aligned}
$$

where $K_{m-1}(1)$ depends on m, α, M, B, and A. We also see that for other k (that can be chosen as $k \leqslant m$), we have $|\Delta^{m-1}_{1/2^n} g_{m-1}(k/2^n)| \leqslant (K_{m-1}(1) + kM)2^{-n\alpha}$. Since $\Delta^m_{1/2^n}\phi(l/2^n) = \Delta^{m-1}_{1/2^n}\phi((l+1)/2^n) - \Delta^{m-1}_{1/2^n}\phi(l/2^n)$, for any ϕ, we have

$$|\Delta^{m-1}_{1/2^n} g_{m-1}((l+1)/2^n)| \leqslant |\Delta^{m-1}_{1/2^n} g_{m-1}(l/2^n)| + |\Delta^m_{1/2^n} g_{m-1}(l/2^n)|,$$

which implies $|\Delta^{m-1}_{1/2^n} g_{m-1}(k/2^n)| \leqslant (K_{m-1}(1) + kM)\, 2^{-n\alpha}$. We continue now to establish C_{m-2} using the same method. We just have to consider now $\Delta^{m-1}_{2^{-n}} g_{m-1}(k/2^n)$, for $k \leqslant m-1$, and define $g_{m-2}(x) = f(x) - (C_{m-1}/(m-1)!)\, x^{m-1} - (C_{m-2}/(m-2)!)\, x^{m-2}$. The constant C_{m-2} is bounded by a constant that depends on m, α, M, B, and A like K of our theorem. Also $\Delta^{m-1}_{1/2^n} g_{m-2}(x) = \Delta^{m-1}_{1/2^n} g_{m-1}(x)$ and $|\Delta^{m-2}_{1/2^n} g_{m-2}(0)| \leqslant K_{m-2}(1)\, 2^{-n\alpha}$, with $K_{m-2}(1)$ depending on the same constants as K_{m-1} and $K_{m-1}(1)$. Moreover,

$$|\Delta^{m-2}_{1/2^n} g_{m-2}(k/2^n)| \leqslant (K_{m-2}(1) + m(K_{m-1}(1) + mM))\, 2^{-n\alpha}, \qquad \text{for} \quad k \leqslant m.$$

We continue and finally obtain $g_0(x) \equiv f(x) - \sum_{l=0}^{m-1} C_l (1/l!)\, x^l$, and $g_0(x)$ satisfies $|\Delta^m_{1/2^n} g_0(k/2^n)| = |\Delta^m_{1/2^n} f(k/2^n)| \leqslant M2^{-n\alpha}$ and $|\Delta^l_{1/2^n} g_0(k/2^n)| \leqslant K2^{-n\alpha}$, for $0 \leqslant l < m$ and $k \leqslant m$. We now define the new function $g_*(x)=g_0(x)$, for $0 \leqslant x < A$, and $g_*(x)=0$, for $x<0$. We now examine $\Delta^m_{1/2^n} g_*(k/2^n)$; for $k \geqslant 0$ the estimate is known; and for $k<0$,

$$|\Delta^m_{1/2^n} g_*(k/2^n)| \leqslant \sum_{l=0}^{m} \binom{m}{l} | g_*(l+k/2^n)| \leqslant 2^m \max_{i \leqslant m-1} | g_*(i/2^n)|$$
$$\leqslant 2^m K_0 2^{-n\alpha}.$$

We make a final adjustment to our constants and function writing $g(x)=g_*(x)+\sum_{l=0}^{m-1} C_l(1/l!)\, x^l$ in $[-A, A]$ and complete our theorem for the case $m-1<\alpha \leqslant m$.

If we do not have $m-1<\alpha\leqslant m$ or, in other words, $\alpha<m-1$ (α cannot be an integer in this case), then we show that $|\Delta^{m-1}_{2^{-n}} f(k/2^n)| \leqslant K2^{-n\alpha}$, for $0 \leqslant k < k_0$. This is shown by a Marchaud-type proof, but here we have an added difficulty for k which is odd, that is, not a multiple of 2. We can write $\Delta^{m-1}_{2^{-n}} f(k/2^n) = \Delta^{m-1}_{2^{-n}}(2[k/2]/2^n)$, for even k, or $\Delta^{m-1}_{2^{-n}} f(k/2^n) = -\Delta^{m-1}_{2^{-n}} f(2[k/2]/2^n) + \Delta^m_{2^{-n}} f(2[k/2]/2^n)$, for odd k. Estimating $\Delta^{m-1}_{2^{-n}} f(2[k/2]/2^n)$, we have

$$\Delta^{m-1}_{2^{-n}} f(2[k/2]/2^n)$$
$$= 2^{-m+1} \Delta^{m-1}_{2^{-n+1}} f([k/2]/2^{n-1})$$
$$+ 2^{-m+1} \sum_{l=0}^{m-2} \sum_{\mu=\nu+1}^{m-1} \binom{m-1}{\mu} \Delta^m_{2^{-n}} f(([k/2]\,2+\nu)\,2^{-n}).$$

Therefore

$$|\Delta^{m-1}_{2^{-n}} f(k/2^n)| \leqslant 2^{-m-1} \max_{0 \leqslant k \leqslant 2^{n-1}-m+1} |\Delta^{m-1}_{2^{-n+1}} f(k/2^{n-1})|$$
$$+ \left(\frac{m-1}{2}+1\right) \max_{0 \leqslant k \leqslant 2^n-m} |\Delta^m_{2^{-n}} f(k/2^n)|,$$

and this inequality would lead to $|\Delta^{m-1}_{2^{-n}} f(k/2^n)| \leqslant M_1 2^{-n\alpha}$, if $m-1>\alpha$, following the proof of Marchaud's inequality (see, for instance, [11, pp. 105–106]).

Actually, the fact that the sequence was 2^{-n} of $K \cdot 2^{-n}$ can be changed to Kl^{-n}, l integer, for any integer, and we obtain the following theorem.

THEOREM 6.2. *If* $h_n = Cl^{-n}$, $f \in C[0, A]$, and $|\Delta^m_{h_n} f(kh_n)| \leqslant Mh_n^\alpha$ *in* $[0, A]$, *and either* $\alpha = m$ *or* α *is not an integer, then we can extend* f *to a function* g *such that* $g(x)=f(x)$ *in* $[0, A]$, $g(x)$ $C[-A, A]$, *and* $|\Delta^m_{h_n} g(kh_n)| \leqslant Mh_n^\alpha$ *for* $[kh_n, (k+m)\,h_n] \subset [-A, A]$.

For the proof which does not require much more than the proof of 6.1 but is much messier, we use Timan [11, p. 103]:

$$\Delta^m_{lh} f(x) = \sum_{\nu_1=0}^{l-1} \cdots \sum_{\nu_m=0}^{l-1} \Delta^m_h f(x+\nu_1 h + \cdots + \nu_m h),$$

and therefore

$$\begin{aligned}\Delta^m_{lh} f(x) - l^m \Delta^m_h f(x) &= \sum_{\nu_1=0}^{l-1} \cdots \sum_{\nu_m=0}^{l-1} [\Delta^m_h f(x+\nu_1 h+\cdots+\nu_m h) - \Delta^m_h f(x)] \\ &= \sum_{\nu_1=0}^{l-1} \cdots \sum_{\nu_m=0}^{l-1} \sum_{t=0}^{\nu_1+\cdots+\nu_m-1} \Delta^{m+1}_h f(x+th).\end{aligned}$$

Substituting $x = Ck/l^n$ and $h = C/l^n$, we can follow the steps of the proof almost word for word, only the formulae will be substantially more complicated.

7. Moduli of Continuity in an Interval and on R^+

We can now deduce the result in an interval or in $\mathbb{R}^+$.

Theorem 7.1. *If $f \in C(R^+)$, $h_n = Cl^{-n}$, for some integer l, and*

$$|\Delta^m_{h_n} f(kh_n)| \leqslant Ml^{-n\alpha}, \qquad \text{for} \quad h = 0, 1, 2, \ldots,$$

where either $\alpha = m$ or α is not an integer, then $|\Delta^m_h f(x)| = M_1 h^\alpha$, for all $h \geqslant 0$ and $x \geqslant 0$.

Theorem 7.2. *If $f(x) \in C[0, 1]$ and $|\Delta^m_{l^{-n}} f(kl^{-n})| \leqslant Ml^{-n\alpha}$, for $k = 0, \ldots, l^n - m$, and either $\alpha = m$ or α is not an integer, then $|\Delta^m_h f(x)| \leqslant M_1 h^\alpha$, for all x and h such that $[x, x+mh] \subset [0, 1]$.*

Proof. These are immediate corollaries of the extension theorems in Section 6 and theorems of Section 3 as well as Theorem 4.7.

Remark. When we have $[a, b]$ instead of $[0, 1]$, we just use $\xi_n = a$ and $h_n = ((b-a)/r)\, l^{-n}$, where r and l are integers.

8. A Result on Locally Lebesgue Integrable $f(x)$

In some sense we restrict our function with an a priori restriction of at least $f \in C(I)$. We will show as a corollary a condition on locally Lebesgue integrable functions that will imply $f \in \text{Lip}\,\alpha$ in C.

Define $a_k(\xi, h) = (1/h) \int_{\xi + kh}^{\xi + (k+1)h} f(u)\, du$ and $\Delta a_k(\xi, h) = a_{k+1}(\xi, h) - a_k(\xi, h)$, while $\Delta^r a_k(\xi, h) = \Delta(\Delta^{r-1} a_k(\xi, h))$.

THEOREM 8.1. *For locally Lebesgue integrable function $f(x)$, any sequence ξ_n and a sequence h_n satisfying $1 \leqslant h_n/h_{n+1} \leqslant M$, the condition $|\Delta^m a_k(\xi_n, h_n)| \leqslant K h_n^\alpha$, for $\alpha \leqslant m$, implies $|f(x+h) - f(x)| \leqslant K h^\alpha$ (for local L_1 equivalent to f).*

Proof. Define $F(x) = \int_0^x f(u)\, du$ and we have $|\Delta^m a_k(\xi_n, h_n)| = |(1/h_n) \Delta_{h_n}^{m+1} F(\xi_n + kh_n)| \leqslant K h_n^\alpha$ or $|\Delta_{h_n}^{m+1} F(\xi_n + kh_n)| \leqslant K h_n^{\alpha+1}$, but $F(x)$ is locally absolutely continuous and therefore, using Theorem 4.6, $|\Delta_h^{m+1} F(x)| \leqslant K_1 h^{\alpha+1}$. This implies that $F'(x) = f(x)$ a.e. and F' is continuous, which implies $|\Delta_h^m f_1(x)| \leqslant M h^\alpha$, for f_1 which is equivalent to $f(x)$.

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